Three-dimensional gravity from SU(2) Yang-Mills theory in two dimensions

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We argue that two-dimensional classical SU(2) Yang-Mills theory describes the embedding of Riemann surfaces in three-dimensional curved manifolds. Specifically, the Yang-Mills field strength tensor computes the Riemannian curvature tensor of the ambient space in a thin neighborhood of the surface. In this sense the two-dimensional gauge theory then serves as a source of three-dimensional gravity. In particular, if the three-dimensional manifold is flat it corresponds to the vacuum of the Yang-Mills theory. This implies that all solutions to the original Gauss-Codazzi surface equations determine two-dimensional integrable models with a SU(2) Lax pair. Furthermore, the three-dimensional SU(2) Chern-Simons theory describes the Hamiltonian dynamics of two-dimensional Riemann surfaces in a four-dimensional flat space-time.

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I. INTRODUCTION

The isometric embedding of a two-dimensional Riemann surface in a three-dimensional ambient space is a classic problem in differential geometry [1]. The embedding involves the first and second fundamental forms of the surface, engaging its intrinsic metric and extrinsic curvature. Together the two fundamental forms determine the metric of the three dimensional ambient space, in a vanishingly thin neighborhood around the two-dimensional surface. If the three dimensional space is flat R^3 the two-dimensional metric and curvature are subject to the original version of the Gauss-Codazzi equations. But when these equations are invalid the three-dimensional curvature is nontrivial, at least in an immediate vicinity of the surface. In this sense the two-dimensional surface is then a source of gravity in the three-dimensional ambient space.

In the present article we shall assert that similarly the two-dimensional SU(2) Yang-Mills field can be viewed as an origin of three-dimensional Riemannian curvature. Indeed, there are suggestions that something like this could occur. For example, it is well known that solutions to the sine-Gordon equation

$$\omega_{tt}-\omega_{xx}+\sin\omega=0 \tag{1}$$

describe constant negative Gaussian curvature surfaces in R^3, up to rigid Euclidean motions. But the sine-Gordon equation also emerges as the zero curvature condition for the SU(2) Yang-Mills field strength tensor [2] (in the sequel $i,j,k,\ldots=1,2$ and $\alpha,\beta,\gamma,\ldots=1,2,3$)

$$F_{ij}^{\alpha}=\partial_i A_j^{\alpha}-\partial_j A_i^{\alpha}+\epsilon^{\alpha\beta\gamma}A_i^{\beta}A_j^{\gamma}=0 \tag{2}$$

when we decompose the connection A_i^{α} according to [2]

$$A_1^{\alpha}\tau^{\alpha}=\frac{\omega_x}{2}\tau^3+k_2\sin\frac{\omega}{2}\tau^1-k_1\cos\frac{\omega}{2}\tau^2$$

$$A_2^{\alpha}\tau^{\alpha}=\frac{\omega_t}{2}\tau^3+k_1\sin\frac{\omega}{2}\tau^1-k_2\cos\frac{\omega}{2}\tau^2, \tag{3}$$

where $k_1^2-k_2^2=1$ and k_i involve the spectral parameter, real for a SU(2) connection. Similar relations between the zero curvature condition (2) in the two-dimensional SU(2) Yang-Mills theory and the embedding of a Riemann surface in flat R^3 have been established for a number of additional integrable models [3].

In the present article we shall argue that for SU(2) the condition (2) always describes the embedding of a Riemann surface in R^3. For this we shall consider a scrupulous decomposition of A_i^{α} which reveals that the condition (2) coincides with the Gauss-Codazzi surface equations that govern the isometric embedding of a Riemann surface in flat R^3, up to rigid rotations and translations. Consequently any integrable model with a SU(2) Lax pair always admits an interpretation in terms of a Riemann surface which is isometrically embedded in the flat three-dimensional Euclidean space.

Furthermore, we shall employ our decomposition of the two-dimensional A_i^{α} to establish that whenever the condition (2) fails, the Yang-Mills field strength tensor F_{ij}^{α} leads to a non-trivial three-dimensional Riemannian curvature tensor in a thin neighborhood of the two-dimensional hypersurface. This implies that the two-dimensional Yang-Mills theory induces gravity in three dimensions. The tentative consistency of this proposal can be verified by comparing the number of field degrees of freedom: The two-dimensional SU(2) gauge field A_i^{α} has six components, which are subject to three Gauss law constraints. Similarly, the three-dimensional metric $G_{\mu\nu}$ ($\mu,\nu=1,2,3$) has six independent matrix elements, and these are amenable to diffeomorphisms which involve three-field degrees of freedom. Thus both two-dimensional A_i^{α} and three-dimensional $G_{\mu\nu}$ carry the same number of field degrees of freedom, with an equal number of gauge degrees of freedom.

We shall now proceed to establish the relationship between the two-dimensional gauge theory and three-dimensional gravity beyond such a simple counting of field degrees freedom. We show that the two-dimensional Yang-

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Mills field strength tensor actually computes the three-dimensional Riemannian curvature tensor in the immediate vicinity of the two-dimensional hypersurface.

II. DECOMPOSITION

The decomposition of vectors and tensors in terms of their irreducible components is a common problem in physics. For example, it is widely employed in fluid dynamics where the velocity three vector decomposes into its gradient and vorticity components. In classical electrodynamics the four dimensional Maxwellian field strength tensor $F_{\mu\nu}$ becomes similarly dissected into its electric and magnetic components. In the context of integrable models, the Lax pair representation leads to decompositions (3) of the two-dimensional non-abelian gauge field in terms of variables that describe the integrable model. Finally, the isometric embedding of a two-dimensional surface (in general d-dimensional hypersurface) with local coordinates y^i in a three-dimensional (in general $d+n$ dimensional) ambient space with local coordinates x^μ and metric $G_{\mu\nu}$, involves the decomposition of the induced metric

$$ds^2 = g_{ij} dy^i dy^j = G_{\mu\nu} \partial_i x^\mu \partial_j x^\nu dy^i dy^j.$$

This embedding also entails the decomposition (Gauss equation)

$$\partial_{ij} x^\mu + \hat{\Gamma}^\mu_{\nu\rho} \partial_i x^\nu \partial_j x^\rho = \Gamma^k_{ij} \partial_k x^\mu + Q_{ij} N^\mu. \tag{4}$$

Here $\hat{\Gamma}^\mu_{\nu\rho}$ is the metric connection in the three-dimensional ambient space, Γ^k_{ij} is the (induced) metric connection on the two-dimensional hypersurface and Q_{ij} is its extrinsic curvature tensor, and N^μ is the three-dimensional unit normal of the hypersurface.

Here we inspect how Eq. (4) relates to the following decomposition of the two-dimensional SU(2) Yang-Mills gauge field A_i^α, introduced originally in [4]

$$A_i^\alpha = C_i n^\alpha + \epsilon^{\alpha\beta\gamma} \partial_i n^\beta n^\gamma + \rho \partial_i n^\alpha + \sigma \epsilon^{\alpha\beta\gamma} \partial_i n^\beta n^\gamma \tag{5}$$

with ρ, σ scalar fields. Notice that we separate the second and fourth term on the right-hand side (RHS) and the reason for this becomes evident as we proceed.

We shall argue that in two dimensions Eq. (5) is a complete decomposition of the full SU(2) Yang-Mills gauge field into its irreducible components.

Furthermore, we shall argue that Eq. (5) admits a differential geometric interpretation in terms of the quantities on the RHS of Eq. (4), describing the isometric embedding of a Riemann surface in a three-dimensional ambient space. Specifically, the vector C_i relates to the (induced) metric connection Γ^k_{ij} on the two-dimensional hypersurface, and ρ and σ relate to the two eigenvalues of its (symmetric) extrinsic curvature tensor Q_{ij}, and the (three component) unit vector n^α maps to the (three component) unit normal N^μ by

$$n^\alpha = e^\alpha{}_\mu N^\mu$$

with $e^\alpha{}_\mu$ a (flat) dreibein that relates the two-unit vectors; these vectors both reside in a flat R^3, the vector n^α is in the tangent bundle of the gauge group SU(2) while N^μ is a vector field in the ambient R^3, the normal map of the two-dimensional hypersurface. These two spaces become identified by e^α_μ which is obviously an element of SO(3).

Finally, we shall argue that when we substitute the decomposition (5) in the Yang-Mills field strength tensor F_{ij}^α it produces the Riemann curvature tensor of the ambient space, when evaluated in the immediate vicinity of the surface.

The decomposition (5) was introduced and inspected in [4], in connection of four-dimensional SU(2) Yang-Mills theory where it is known to be incomplete [5]. But we now argue that in two dimensions the decomposition (5) is complete, describing the six independent components of a generic two-dimensional SU(2) gauge field A_i^α.

Indeed, when $D=2$ the vector field C_i has two components. Together with ρ and σ and the two independent components of the unit vector n^α, both sides of Eq. (5) engage six-field degrees of freedom.

In order to confirm that the six field degrees of freedom on the RHS of Eq. (5) are actually independent, we first substitute the decomposition in the Yang-Mills field strength tensor. This yields

$$\begin{aligned} F_{ij}^\alpha = {} & (G_{ij} - [1-(\rho^2+\sigma^2)]H_{ij}) n^\alpha + \nabla_i \rho \partial_j n^\alpha \\ & + \nabla_i \sigma \epsilon^{\alpha\beta\gamma} \partial_j n^\beta n^\gamma - \nabla_j \rho \partial_i n^\alpha - \nabla_j \sigma \epsilon^{\alpha\beta\gamma} \partial_i n^\beta n^\gamma. \end{aligned} \tag{6}$$

Here

$$G_{ij} = \partial_i C_j - \partial_j C_i H_{ij} = \epsilon^{\alpha\beta\gamma} n^\alpha \partial_i n^\beta \partial_j n^\gamma$$

and

$$(\partial_i + i C_i)(\rho + i\sigma) = \nabla_i (\rho + i\sigma) \equiv \nabla_i \phi.$$

We then substitute this decomposition of F_{ij}^α in the Yang-Mills action

$$S = \frac{1}{4} \int d^2 x (F_{ij}^\alpha)^2. \tag{7}$$

When we perform a variation of this action with respect to the component fields (C_i, ϕ, n^α), the ensuing critical points lead to a set of Euler-Lagrange equations. These equations reproduce the full two-dimensional Yang-Mills equations

$$D_i^{\alpha\beta} F_{ij}^\beta = 0 \tag{8}$$

only when the decomposition (5) is complete in directions which are orthogonal to the gauge orbits

$$A_i^\alpha \to A_i^\alpha + D_i^{\alpha\beta} \epsilon^\beta \equiv A_i^\alpha + (\delta^{\alpha\beta} \partial_i + \epsilon^{\alpha\gamma\beta} A_i^\gamma) \epsilon^\beta. \tag{9}$$

The variation of the action (7) with respect to the components (C_i, ϕ, n^a) gives the following Euler-Lagrange equations [4]

$$\hat{\mathbf{n}} \cdot D_i \mathbf{F}_{ij} = 0,$$

$$\kappa_j^+ \hat{\mathbf{e}}_+ \cdot D_i \mathbf{F}_{ij} = 0,$$

$$\nabla_j \phi \hat{\mathbf{e}}_- \cdot D_i \mathbf{F}_{ij} = 0. \tag{10}$$

Here $(\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\varphi, \hat{\mathbf{n}})$ is a right-handed orthonormal triplet and

$$\kappa_i^+ = \kappa_i^\theta + i\kappa_i^\varphi = (\hat{\mathbf{e}}_\theta + i\hat{\mathbf{e}}_\varphi) \cdot \partial_i \hat{\mathbf{n}} = \hat{\mathbf{e}}_+ \cdot \partial_i \hat{\mathbf{n}}. \tag{11}$$

Note that there is some latitude in the definition of $\hat{\mathbf{e}}_+ = \hat{\mathbf{e}}_\theta + i\hat{\mathbf{e}}_\varphi$, without affecting any of our subsequent conclusions we can send

$$\hat{\mathbf{e}}_+ \to e^{i\xi} \hat{\mathbf{e}}_+ . \tag{12}$$

Thus we have an internal U(1) gauge structure which has been discussed in [5].

Since $F_{12}^\alpha = -F_{21}^\alpha$ we immediately find that the only nontrivial regular solution to the equations (10) is the homogeneous one,

$$\text{Eqs. (10)} \Rightarrow D_i^{\alpha\beta} F_{ij}^\beta = 0$$

that is, the full two-dimensional Yang-Mills equation (8). This means that the decomposition (5) is indeed complete in the space of gauge orbits of A_i^α.

For a total completeness of the decomposition (5) we still need to identify in it the SU(2) gauge orbit (9). This gauge orbit involves three field degrees of freedom. One of these is the U(1) gauge transformation in the direction of n^α, with $\epsilon^\alpha = \epsilon n^\alpha$. It sends

$$C_i \to C_i - \partial_i \epsilon,$$

$$\phi = \rho + i\sigma \to e^{i\epsilon}\phi,$$

while n^α itself remains intact. Notice that as a consequence (C_i, ϕ) has a natural interpretation as an (electric) Abelian Higgs multiplet. Since the unit vector n^α has a natural magnetic interpretation (it appears as an order parameter, e.g., in the Heisenberg model) the two sets of variables (C_i, ϕ) and n^α are inherent electric and magnetic dual variables in the two-dimensional gauge theory.

The remaining two field degrees of freedom along the gauge orbit must be orthogonal to n^α. They can be described as follows: We introduce $g \in \mathrm{SU}(2)$ by

$$n^\alpha \tau^\alpha = g \tau^3 g^{-1},$$

which is manifestly U(1) invariant, i.e., invariant under conjugation $g \to gh$ by an element $h \in SU(2)$ in the Cartan direction τ^3. This corresponds to the U(1) gauge transformation along n^α.

We introduce the right-invariant form

$$R_i = g^{-1} \partial_i g.$$

With R_i^{diag} the diagonal part of R_i and R_i^{off} its off-diagonal part, we can write the gauge field (5) as

$$A_i^\alpha \tau^\alpha = g(C_i \tau^3 + iR_i^{diag} + \rho[R_i, \tau^3] - i\sigma R_i^{off}) g^{-1} + ig\partial_i g^{-1}. \tag{13}$$

Consequently Eq. (5) is manifestly gauge-equivalent to (in the sequel we always have $a,b = 1,2$)

$$B_i^\alpha \tau^\alpha = C_i \tau^3 + iR_i^{diag} + \rho[R_i, \tau^3] - i\sigma R_i^{off} \equiv W_i \tau^3 + Q_i^a \tau^a. \tag{14}$$

This reveals that the parametrization (5), (13) is indeed complete, also on the gauge orbit space.

Clearly, the sine-Gordon decomposition (3) must be contained in Eqs. (13), (14). Comparing Eq. (3) with Eq. (14) we conclude that we must choose ρ, σ and n^α such that

$$k_2 \sin\frac{\omega}{2} - ik_1 \cos\frac{\omega}{2} = (\rho + i\sigma)(\kappa_1^\theta + i\kappa_1^\varphi) \equiv \phi \kappa_1^+$$

$$k_1 \sin\frac{\omega}{2} - ik_2 \cos\frac{\omega}{2} = (\rho + i\sigma)(\kappa_2^\theta + i\kappa_2^\varphi) \equiv \phi \kappa_2^+ . \tag{15}$$

We parametrize

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}, \tag{16}$$

and we select the phase (12) so that Eq. (11) becomes

$$\kappa_i^+ = \kappa_i^\theta + i\kappa_i^\varphi = \partial_i \theta + i \sin\theta \partial_i \varphi.$$

We then get from Eq. (15)

$$\begin{pmatrix} \partial_1 \theta \\ \sin\theta \partial_1 \varphi \end{pmatrix} = \frac{1}{\rho^2 + \sigma^2} \begin{bmatrix} \rho & \sigma \\ -\sigma & \rho \end{bmatrix} \begin{pmatrix} k_2 \sin\dfrac{\omega}{2} \\ k_1 \cos\dfrac{\omega}{2} \end{pmatrix},$$

$$\begin{pmatrix} \partial_2 \theta \\ \sin\theta \partial_2 \varphi \end{pmatrix} = \frac{1}{\rho^2 + \sigma^2} \begin{bmatrix} \rho & \sigma \\ -\sigma & \rho \end{bmatrix} \begin{pmatrix} k_1 \sin\dfrac{\omega}{2} \\ k_2 \cos\dfrac{\omega}{2} \end{pmatrix},$$

from which we can solve θ and φ in terms of ω which is a solution of the sine-Gordon equation, and ρ and σ which we can select quite liberally.

III. THREE-DIMENSIONAL RIEMANN TENSOR

We now proceed to show that the Yang-Mills field strength tensor (6) can be viewed as the Riemannian curvature tensor in a three-dimensional ambient space in the immediate vicinity of a two-dimensional hypersurface. For a suggestive correspondence, we start by interpreting the Yang-Mills field A_i^α as a linear combination of a background field W_i which is the Cartan component in Eq. (14), and a fluctuation field Q_i^a which corresponds to the off-diagonal part in Eq. (14). The Yang-Mills field strength tensor then decomposes into the following Cartan part and off-diagonal part

$$F_{ij}^3=\partial_i W_j-\partial_j W_i+(Q_i^1Q_j^2-Q_j^2Q_i^1)\equiv F_{ij}+(Q_i^1Q_j^2-Q_j^2Q_i^1), \tag{17}$$

$$F_{ij}^a=(\delta^a{}_b\partial_i-W_i\epsilon^a{}_b)Q_j^b-(\delta^a{}_b\partial_j-W_j\epsilon^a{}_b)Q_i^b \quad (a,b=1,2). \tag{18}$$

It is instructive to compare this with Eq. (6), which is the representation of Eqs. (17), (18) in the gauge $\tau^3\to n^\alpha\tau^\alpha$. The structural similarity is evident.

Next, we recall Ricci's identity which states

$$(\nabla_\rho\nabla_\sigma-\nabla_\sigma\nabla_\rho)\partial_\eta x^\kappa=\partial_\tau x^\kappa R^\tau{}_{\eta\rho\sigma} \tag{19}$$

for a connection ∇_ρ and the ensuing curvature tensor $R^\tau{}_{\eta\rho\sigma}$. We employ this in the Gauss equation (4), by selecting for ∇_ρ the induced covariant derivative on the hypersurface. This gives for the Riemann curvature tensor of the ambient space the decomposition

$$\hat{R}^\mu{}_{\nu\rho\sigma}\partial_i x^\nu\partial_j x^\rho\partial_k x^\sigma=\partial_l x^\mu U^l{}_{ijk}+N^\mu V_{ijk}, \tag{20}$$

where

$$U_{lijk}=R_{lijk}+(Q_{ij}Q_{kl}-Q_{ik}Q_{jl}), \tag{21}$$

$$V_{ijk}=(\delta_i^l\partial_k-\Gamma_{ik}^l)Q_{lj}-(\delta_i^l\partial_j-\Gamma_{ij}^l)Q_{lk} \tag{22}$$

with R_{lijk} the Riemann tensor on the two-dimensional surface and Q_{ij} its extrinsic curvature.

Clearly, there is a definite formal similarity between Eqs. (17) and (21), and between Eqs. (18) and (22) suggesting that we can relate $F_{ij}^3\sim U_{lijk}$ and $F_{ij}^a\sim V_{ijk}$. If this identification indeed holds, the two-dimensional Yang-Mills field strength tensor computes the three-dimensional ambient Riemann curvature tensor: The U_{lijk} is the restriction of the Riemann tensor to the tangent of the surface, and V_{ijk} is the projection of the Riemann tensor along the unit normal of the surface. Consequently we obtain the entire three-dimensional Riemann curvature tensor from the two-dimensional F_{ij}^a, in the vicinity of the two-dimensional hypersurface. In this sense the three-dimensional gravity is then induced by the two-dimensional Yang-Mills theory.

We shall now proceed to establish the relations between Eqs. (17), (18) and Eqs. (21), (22). For this we denote by $u,v,\ldots=1,2$ a local frame (tangent bundle) on the two-dimensional hypersurface in the three-dimensional ambient space. The ensuing zweibein obeys $e^u{}_ie^v{}_j\eta_{uv}=g_{ij}$ and $e^u{}_iE^i{}_v=\delta^u{}_v$ etc. Furthermore, we introduce $\epsilon^u{}_v$ with $\epsilon^1{}_2=-\epsilon^2{}_1=1$. We also introduce the zweibein $e^a{}_u$ with inverse $E^u{}_a$, which relate the local frame of the hypersurface to the off-diagonal part of the SU(2) Lie-algebra.

We start with the decomposition (14), where we write

$$Q_i^{1,2}\equiv Q^a{}_i=e^a{}_uQ^u{}_i=e^a{}_ue^u{}_jQ^j{}_i\,.$$

We then consider F_{ij}^a. From Eq. (18) we get

$$\begin{aligned}E^v{}_aF_{ij}^a&=\partial_iQ^v{}_j+(E^v{}_a\partial_ie^a{}_u-W_i\epsilon^v{}_u)Q^u{}_j-(i\leftrightarrow j)\\&=\eta^{vw}E^k{}_w\partial_iQ_{kj}+([E^v{}_a\partial_ie^a{}_u-W_i\epsilon^v{}_u]\eta^{uw}E^k{}_w\\&\quad+\eta^{vw}\partial_iE^k{}_w)Q_{kj}-(i\leftrightarrow j).\end{aligned}$$

Consider

$$E^v{}_a\partial_ie^a{}_u-W_i\epsilon^v{}_u\,.$$

Here $e^a{}_u$ is a zweibein between two-dimensional flat Euclidean spaces, and it can be represented explicitly, e.g., as

$$e^1{}_u=\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}\quad\text{and}\quad e^2{}_u=\begin{pmatrix}-\sin\psi\\ \cos\psi\end{pmatrix},$$

so that

$$E^v{}_a\partial_ie^a{}_u-W_i\epsilon^v{}_u=-(W_i-\partial_i\psi)\epsilon^v{}_u\,.$$

This suggests that we introduce a U(1) gauge transformation in the Cartan direction of SU(2), and redefine

$$W_i-\partial_i\psi\to W_i\,. \tag{23}$$

This gives

$$e^u{}_k\eta_{uw}E^w{}_aF_{ij}^a=\partial_iQ_{kj}-(E^l{}_u\partial_ie^u{}_k-W_i\epsilon^l{}_k)Q_{lj}-(i\leftrightarrow j). \tag{24}$$

We recall the familiar relation between spin connection and Christoffel symbol,

$$\Gamma_{ki}^l=\omega^l{}_{ki}+E^l{}_u\partial_ie^u{}_k\,.$$

Hence, if we identify

$$\omega^l{}_{ki}=-W_i\epsilon^l{}_k \tag{25}$$

we can write Eq. (24) as

$$e^u{}_k\eta_{uv}E^v{}_aF_{ij}^a=[\partial_iQ_{kj}-\Gamma_{ik}^lQ_{lj}]-[\partial_jQ_{ki}-\Gamma_{jk}^lQ_{li}]. \tag{26}$$

Thus

$$F_{ij}^a=e^a{}_u\eta^{uv}E^k{}_vV_{jik} \tag{27}$$

and consequently the off-diagonal part of the two-dimensional Yang-Mills field strength tensor computes the tangential part (21) of the three-dimensional Riemann curvature tensor (20).

We now proceed to inspect the Cartan component F_{ij}^3 of the Yang-Mills field strength tensor. When we recall the representation of the Riemann tensor in terms of the spin connection, we get

$$\begin{aligned}R^u{}_{vij}&=\partial_i\omega^u{}_{vj}-\partial_j\omega^u{}_{vi}+\omega^u{}_{wi}\omega^w{}_{vj}-\omega^u{}_{wj}\omega^w{}_{vi}\\&=-(\partial_iW_j-\partial_jW_i)\epsilon^u{}_v\,,\end{aligned}$$

where we have used Eq. (25). But from Eq. (17) we now get immediately the desired relation

$$F_{ij}^{3}=\frac{1}{2}\epsilon^{v}{}_{u}(R^{u}{}_{vij}-Q^{u}{}_{i}Q_{vj}+Q^{u}{}_{j}Q_{vi})=\frac{1}{2}\epsilon^{v}{}_{u}U^{u}{}_{vij} \quad (28)$$

and we conclude that F_{ij}^3 indeed computes the normal component of the three-dimensional Riemann tensor.

When we combine Eq. (28) with Eq. (27) we arrive at our main result: The two-dimensional Yang-Mills field strength tensor can be interpreted as a three-dimensional Riemann curvature tensor in the vicinity of the two-dimensional hypersurface. In this sense, the two-dimensional Yang-Mills theory is then a source of gravity in the three-dimensional ambient space.

IV. FURTHER DEVELOPMENTS

The present results can be extended in a variety of directions. For example, the flatness condition (2) in the two-dimensional gauge theory can also be interpreted as the equation of motion (first class constraint) in the three-dimensional SU(2) Chern-Simons theory, when viewed as a Hamiltonian system

$$S=\int d^3x\,\mathrm{Tr}\left[A\wedge dA+\frac{2}{3}A^3\right]$$
$$\to\int d^2x\,dt(\epsilon^{ij}A_i^\alpha\partial_t A_j^\alpha-A_0^\alpha\epsilon^{ij}F_{ij}^\alpha). \quad (29)$$

Here we have shown that the condition (2) can also be identified with the Gauss-Codazzi equations, which modulo rigid rotations and translations describe the embedding of two-dimensional Riemann surfaces in flat three-dimensional R^3. Consequently the Chern-Simons theory determines the Hamiltonian dynamics of two-dimensional Riemann surfaces in flat R^3. Since the condition (2) also relates to the Lax pair of integrable models, the dynamics of these Riemann surfaces is integrable, and the surfaces scatter from each other in an elastic manner which directly relates to the properties of conventional two-dimensional integrable models [2]. The SU(2) Chern-Simons theory also describes three-dimensional knot invariants [6] suggesting interesting connections between knot theory and the dynamics of two-dimensional Riemann surfaces in R^3.

Furthermore, since Eq. (2) describes the embedding of Riemann surfaces in a flat three-dimensional space the ensuing Chern-Simons theory does not employ four-dimensional gravity. It would be very interesting to develop a generalization of the Chern-Simons theory, with four-dimensional gravity included. This generalization should lead to Eqs. (20), (27), (28) as its equations of motion, describing the dynamics of two-dimensional Riemann surfaces in four dimensional curved ambient space with a curvature induced by the Riemann surfaces, radiating gravity.

Finally, we note that various other relations between gauge fields and gravity have been studied in many other context. For example the Liouville theory descends from an SL(2,R) gauge theory with appropriate constraints [7]. This is also related to the AdS3/CFT2 correspondence which provides a relation between two-dimensional gauge theories and three-dimensional gravity [8]. Furthermore, the Jackiw-Teitelboim model of two-dimensional gravity can also be written in terms of flat connections [9]. While these and other similar relations have no straightforward connection to the present work, it would be interesting to see how our results can be interpreted in these perspectives.

V. CONCLUSIONS

In conclusion, we have shown that the two-dimensional SU(2) Yang-Mills field strength tensor can be interpreted as a three-dimensional Riemann curvature tensor. This can be further interpreted so that the two-dimensional gauge theory is a source of three-dimensional gravity. A vanishing Yang-Mills field strength tensor then leads to a vanishing Riemannian curvature, and consequently it has an interpretation in terms of the original Gauss-Codazzi equations which describe the isometric embedding of Riemann surfaces in flat R^3. The vanishing Yang-Mills field strength tensor also yields a SU(2) Lax pair which implies that two-dimensional integrable models with such a Lax pair specify Riemann surfaces in flat R^3. Furthermore, since a vanishing two-dimensional field strength tensor also arises as the Hamiltonian equation of motion in three-dimensional Chern-Simons theory, this theory admits an interpretation in terms of Hamiltonian dynamics of two-dimensional Riemann surfaces in flat four dimensional ambient space. Obviously it would be interesting to generalize the Chern-Simons theory so that it allows for a nontrivial four dimensional curvature.

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